Binary Relations

Part II

Outline for Today

- ***Finish from Last Time***
  - Pts. 2, 3 of our proof that ~ is an equivalence relation
- ***Properties of Equivalence Relations***
  - What's so special about those three rules?
- ***Cyclic Property***
  - How it relates to our other three properties, and equivalence relations

Finish from Last Time

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

An Example Relation

- Consider the binary relation ~ defined over the set ℤ:

$$a \sim b \quad \textbf{if} \quad a+b \textbf{ is even}$$

- Some examples:

$$0\sim4 \qquad 1\sim9 \qquad 2\sim6 \qquad 5\sim5$$

- Turns out, this is an equivalence relation! Let's see how to prove it.

We can binary relations by giving a rule, like this:

$$a \sim b \quad \textbf{if} \quad \textit{some property of a and b holds}$$

This is the general template for defining a relation. Although we're using “if” rather than “iff” here, the two above statements are definitionally equivalent. For a variety of reasons, definitions are often introduced with “if” rather than “iff.” Check the “Mathematical Vocabulary” handout for details.

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation $\sim$ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the $\sim$ relation, this means that we need to prove that $a+a$ is even.

To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer k (namely, a), so $a+a$ is even. Therefore, $a \sim a$ holds, as required. ■

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

***$a \sim b$* if *$a+b$* is even**

Lemma 2: The binary relation $\sim$ is symmetric.

Which of the following works best as the opening of this proof?

A. Consider any integers a and b. We will prove $a \sim b$ and $b \sim a$.
B. Pick $\forall a \in \mathbb{Z}$ and $\forall b \in \mathbb{Z}$. We will prove $a \sim b \to b \sim a$.
C. Consider any integers a and b where $a \sim b$ and $b \sim a$.
D. Consider any integer a where $a \sim a$.
E. The relation $\sim$ is symmetric if for any $a, b \in \mathbb{Z}$, we have $a \sim b \to b \sim a$.
F. Consider any integers a and b where $a \sim b$. We will prove $b \sim a$.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, **D**, **E**, or **F**.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof:

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation ~ is symmetric.

Proof:

What is the formal definition of symmetry?

$$a \sim b \quad \text{if} \quad a+b \text{ is even}$$

Lemma 2: The binary relation ~ is symmetric.

Proof:

What is the formal definition of symmetry?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ (a \sim b \to b \sim a)$$

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof:

What is the formal definition of symmetry?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ (a \sim b \to b \sim a)$$

Therefore, we'll choose arbitrary integers a and b where $a \sim b$, then prove that $b \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation ~ is symmetric.

Proof:

What is the formal definition of symmetry?

$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ (a \sim b \to b \sim a)$

Therefore, we'll choose arbitrary integers a and b where $a \sim b$, then prove that $b \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation ~ is symmetric.

Proof:

What is the formal definition of symmetry?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ (a \sim b \rightarrow b \sim a)$$

Therefore, we'll choose arbitrary integers a and b where $a \sim b$, then prove that $b \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers a and b where $a \sim b$.
We need to show that $b \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers a and b where $a \sim b$.
We need to show that $b \sim a$.

Which of the following works best as the opening of this proof?

A. Consider any integers a and b. We will prove $a \sim b$ and $b \sim a$.
B. Pick $\forall a \in \mathbb{Z}$ and $\forall b \in \mathbb{Z}$. We will prove $a \sim b \to b \sim a$.
C. Consider any integers a and b where $a \sim b$ and $b \sim a$.
D. Consider any integer a where $a \sim a$.
E. The relation $\sim$ is symmetric if for any $a, b \in \mathbb{Z}$, we have $a \sim b \to b \sim a$.
F. Consider any integers a and b where $a \sim b$. We will prove $b \sim a$.

$a{\sim}b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers a and b where $a{\sim}b$.
We need to show that $b{\sim}a$.

Since $a{\sim}b$, we know that $a+b$ is even.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation ~ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even. Since $b+a$ is even, we know that $b \sim a$, as required.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even. Since $b+a$ is even, we know that $b \sim a$, as required. ■

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Which of the following works best as the introduction to this proof?

A. Pick an arbitrary a and b from A where $a \sim b$. We'll prove $b \sim a$.
B. Consider any $a, b, c \in A$ where $a \sim b$, $b \sim c$, and $a \sim c$.
C. Choose an $a, b, c \in A$. We will prove $a \sim b$, $b \sim c$, and $a \sim c$.
D. Take any $a, b, c \in A$ where $a \sim b$ and $b \sim c$; we'll prove $a \sim c$.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, or **D**.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof:

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof:

What is the formal definition of transitivity?

$$a \sim b \quad \textbf{if} \quad a+b \textbf{ is even}$$

Lemma 3: The binary relation ~ is transitive.

Proof:

What is the formal definition of transitivity?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ \forall c \in \mathbb{Z}.\ (a \sim b \land b \sim c \to a \sim c)$$

$a \sim b$ **if** $a + b$ **is even**

Lemma 3: The binary relation ~ is transitive.

Proof:

What is the formal definition of transitivity?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ \forall c \in \mathbb{Z}.\ (a \sim b \land b \sim c \to a \sim c)$$

Therefore, we'll choose arbitrary integers a, b, and c where $a \sim b$ and $b \sim c$, then prove that $a \sim c$.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even.

***a*∼*b* if *a*+*b* is even**

Lemma 3: The binary relation ∼ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so

$$a+c = 2k + 2m - 2b = 2(k+m-b).$$

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so

$$a+c = 2k + 2m - 2b = 2(k+m-b).$$

So there is an integer r, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so

$$a+c = 2k + 2m - 2b = 2(k+m-b).$$

So there is an integer r, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required. ■

***a*~*b*   if   *a*+*b* is even**

Lemma 3: The binary relation ~ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so

$$a+c = 2k +$$

So there is an integer r,

$a+c = 2r$. Thus $a+c$ is ev

The formal definition of transitivity is given in first-order logic, but **this proof does not contain any first-order logic symbols!**

First-Order Logic and Proofs

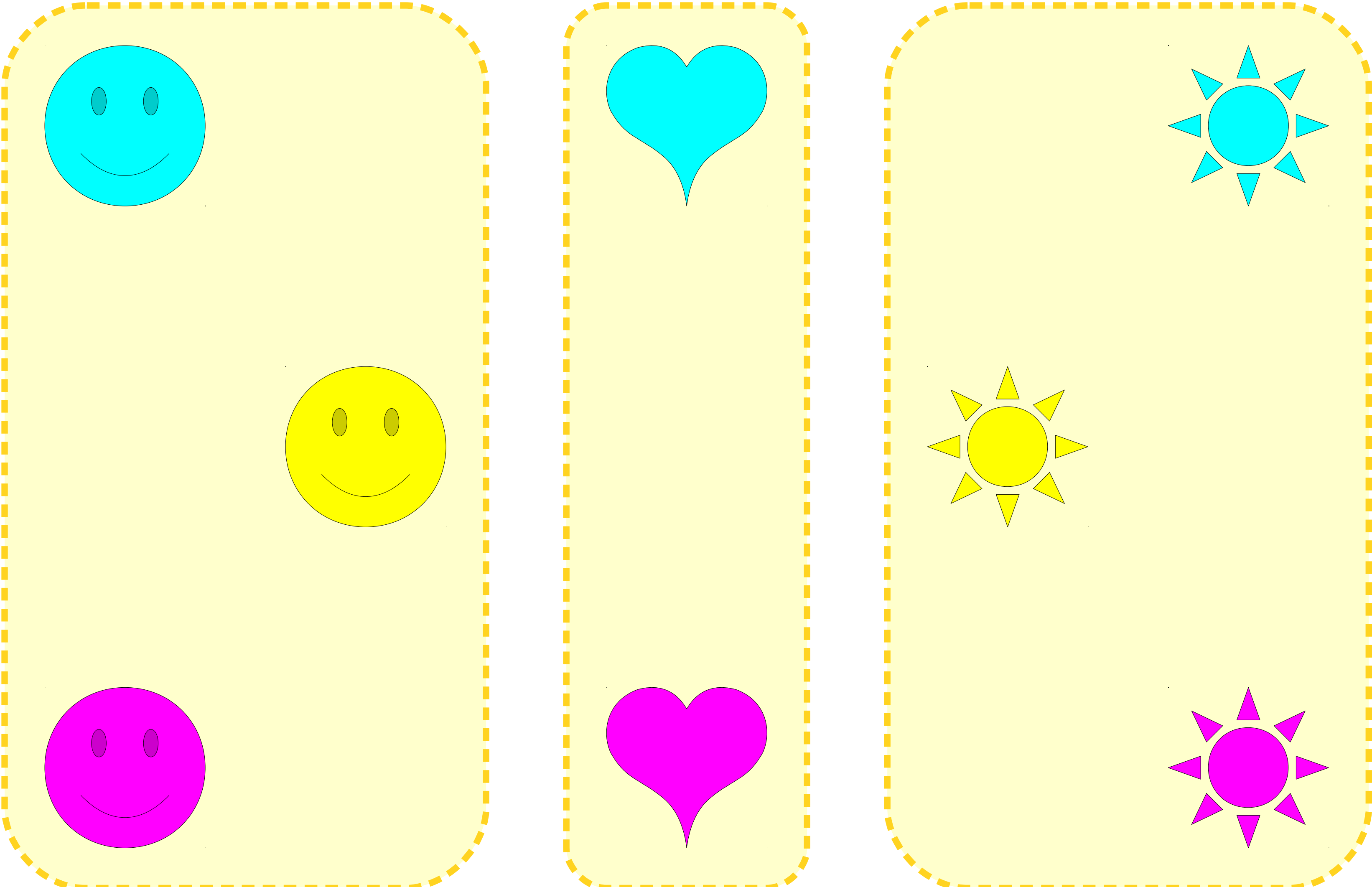
- First-order logic is an excellent tool for giving formal definitions to key terms.
- While first-order logic *guides* the structure of proofs, it is *exceedingly rare* to see first-order logic in written proofs.
- Follow the example of these proofs:
  - Use the FOL definitions to determine what to assume and what to prove.
  - Write the proof in plain English using the conventions we set up in the first week of the class.
- ***Please, please, please, please, please internalize the contents of this slide!***

$$\forall a \in A.\ aRa$$

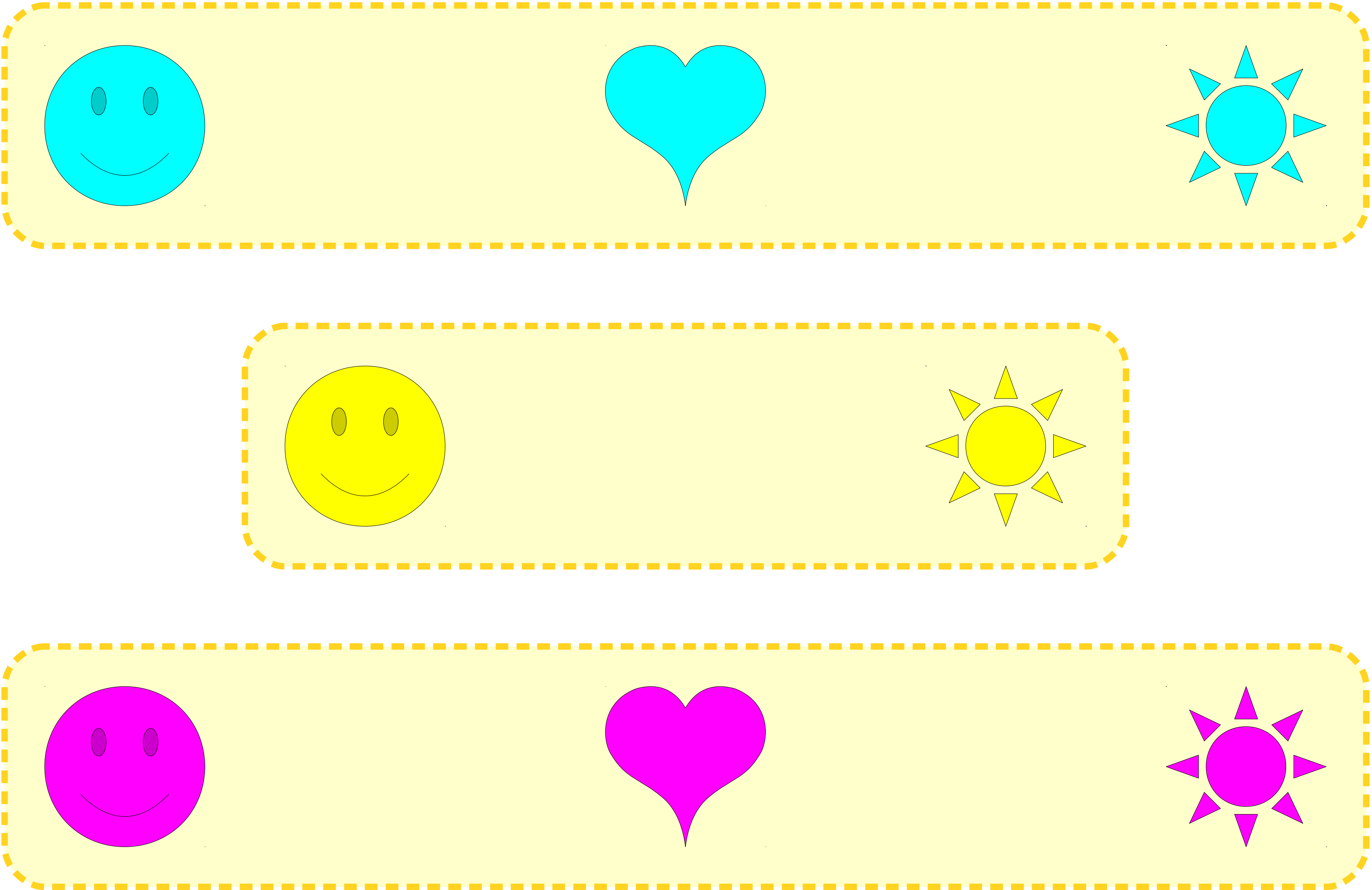
$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

Properties of Equivalence Relations



xRy if x and y have the same shape



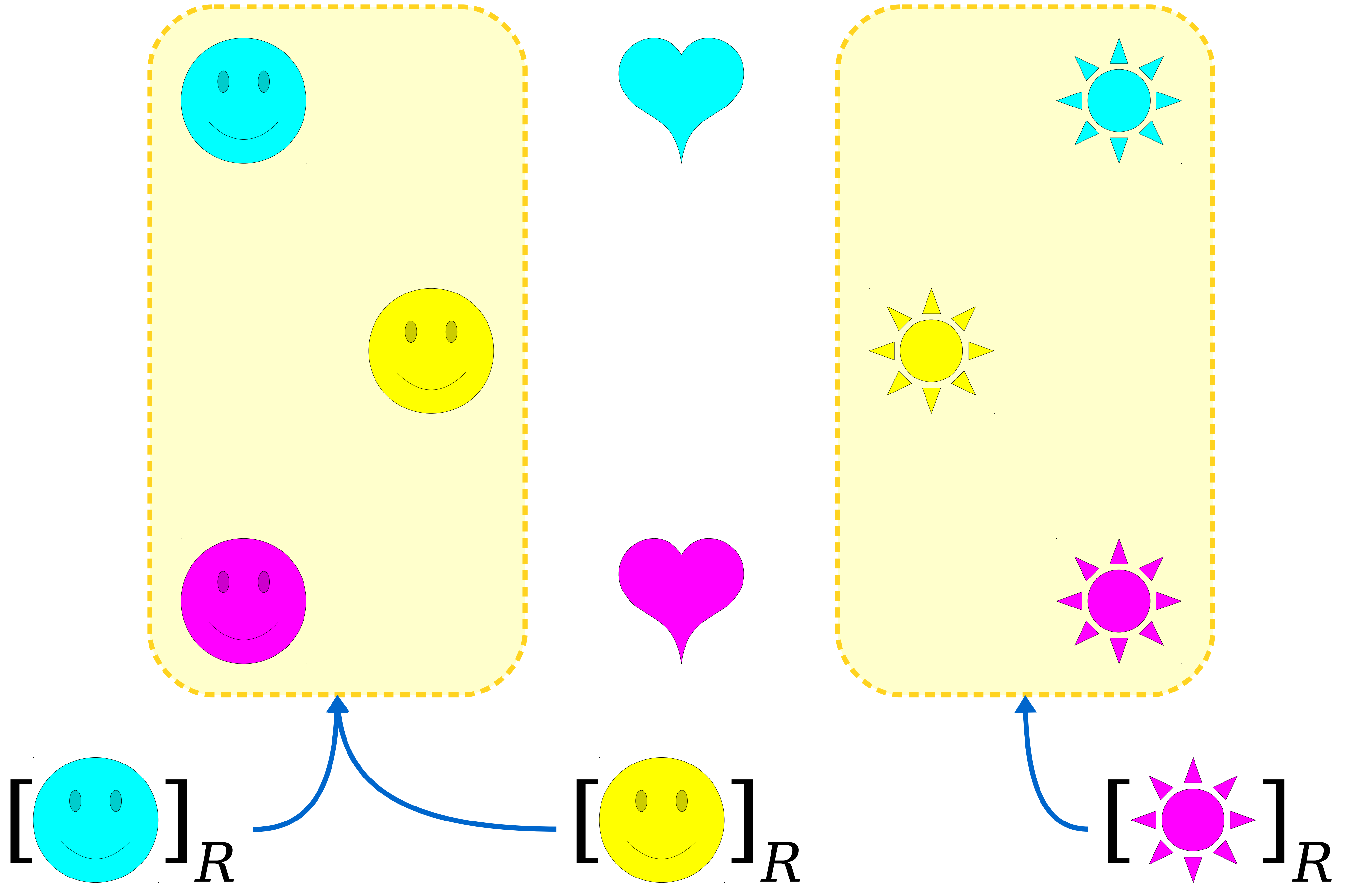
xTy if x and y have the same color

Equivalence Classes

- Given an equivalence relation R over a set A, for any $x \in A$, the ***equivalence class of x*** is the set

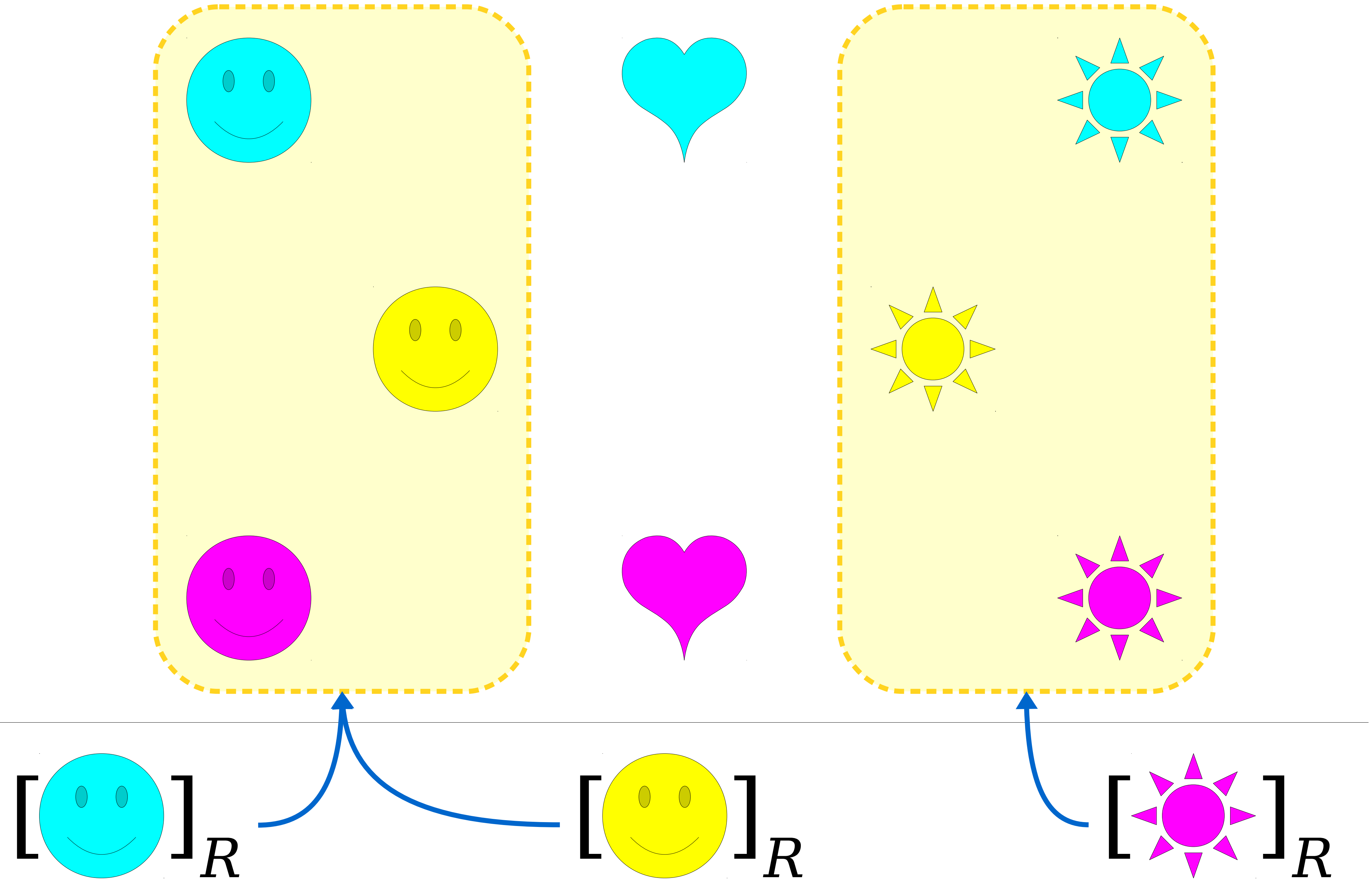
$$[x]_R = \{\ y \in A \mid xRy\ \}$$

- Intuitively, the set $[x]_R$ contains all elements of A that are related to x by relation R.



xRy if x and y have the same shape

The Fundamental Theorem of Equivalence Relations: Let R be an equivalence relation over a set A. Then every element $a \in A$ belongs to exactly one equivalence class of R.

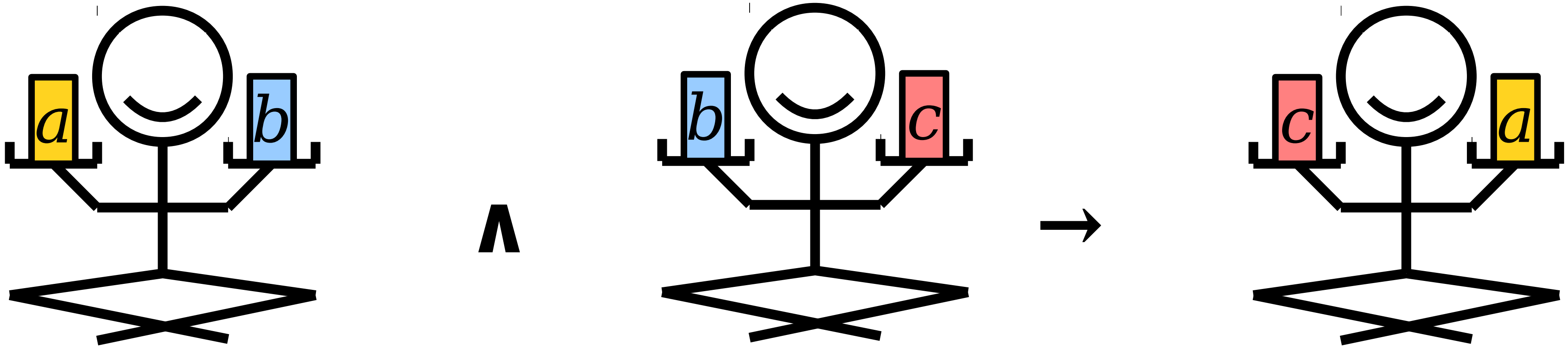


xRy if x and y have the same shape

How'd We Get Here?

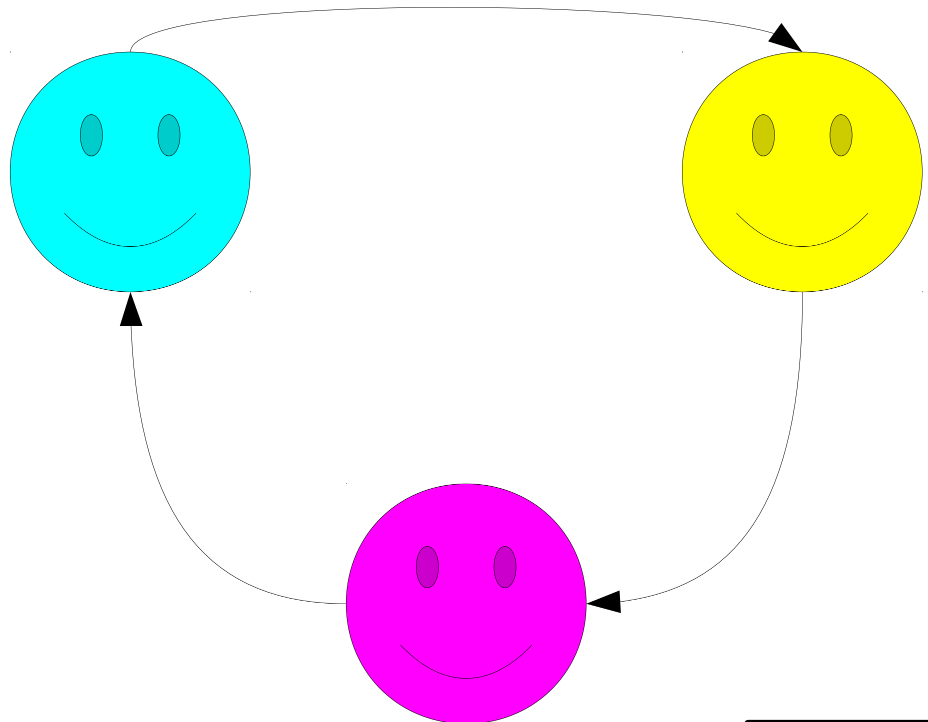
- We discovered equivalence relations by thinking about ***partitions*** of a set of elements.
- We saw that if we had a binary relation that tells us whether two elements are in the same group, it had to be reflexive, symmetric, and transitive.
- The FToER says that, in some sense, these rules precisely capture what it means to be a partition.
- ***Question:*** What's so special about these three rules?

The question we are asking the sage: “Are these two in the same equivalence class?”



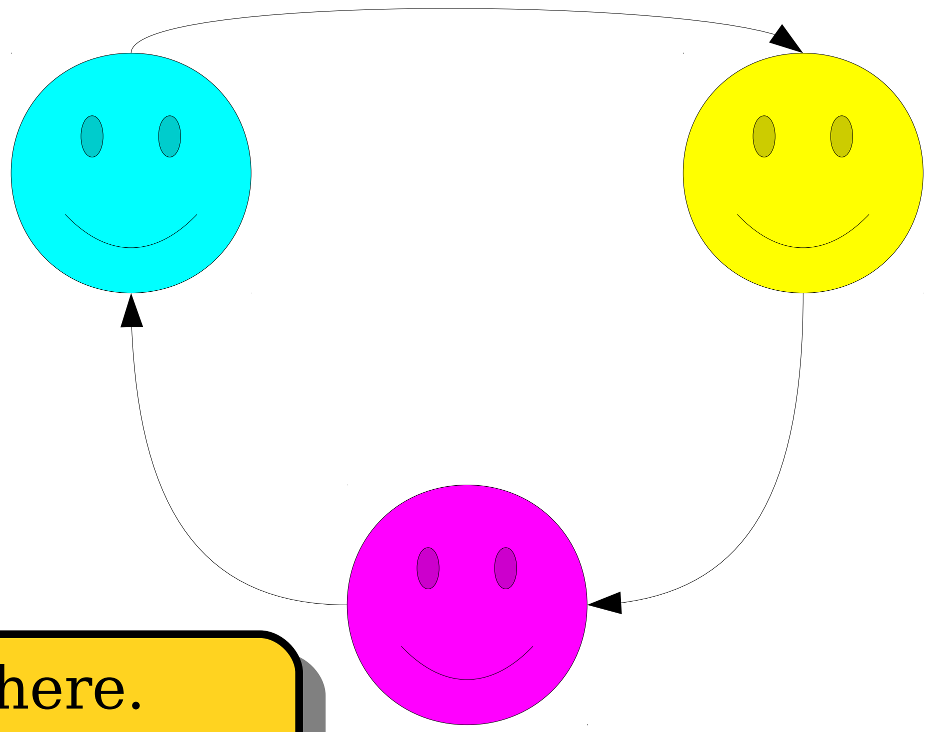
$$aRb \quad \wedge \quad bRc \quad \rightarrow \quad cRa$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$



A binary relation with this property is called ***cyclic***.

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$



Let R be the relation depicted here.
How many of the following claims are true?

R is reflexive.
R is symmetric.
R is transitive.
R is an equivalence relation.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, or **4**.

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$

Theorem: A binary relation R over a set A is an equivalence relation if and only if it is reflexive and cyclic.

Theorem: A binary relation R over a set A is an equivalence relation if and only if it is reflexive and cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

What We Need To Show

- R is reflexive.
- R is cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming

R is an equivalence relation.

- R is reflexive.

  R is symmetric.

  R is transitive.

What We Need To Show

- R is reflexive.

  R is cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

What We Need To Show

- R is reflexive.
- R is cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

What We Need To Show

R is reflexive.

R is cyclic.

- If aRb and bRc, then cRa.

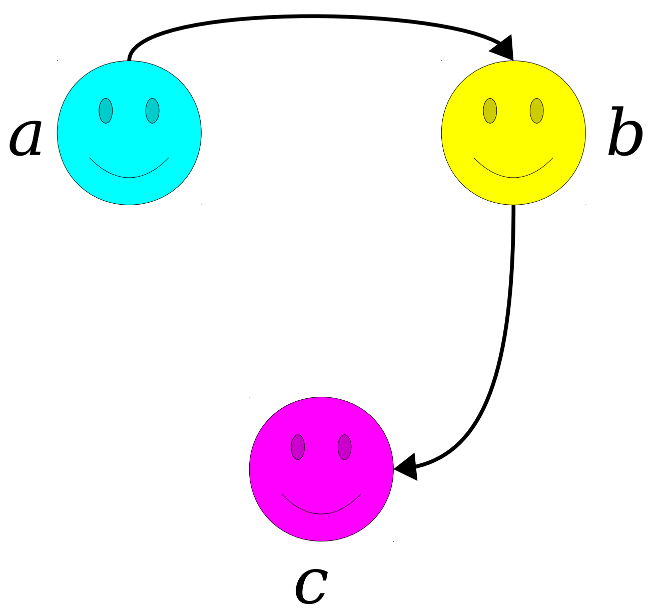
Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

What We Need To Show

- If aRb and bRc, then cRa.



Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

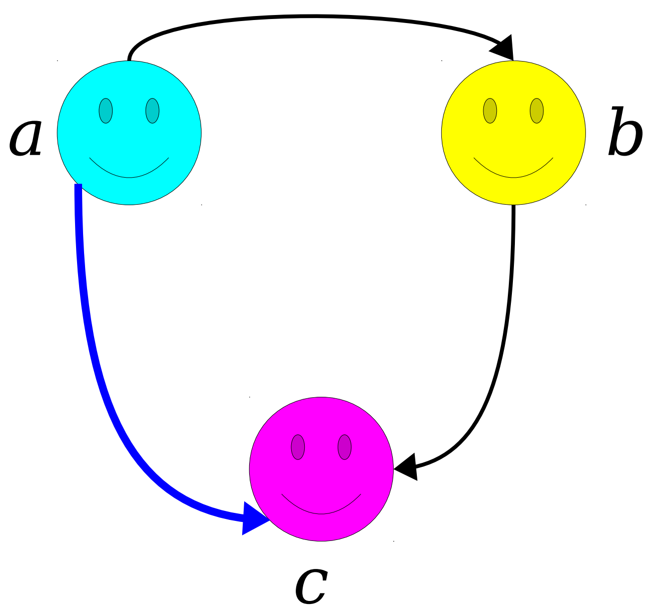
What We're Assuming

R is an equivalence relation.

- R is reflexive.
- R is symmetric.
- R is transitive.

What We Need To Show

- If aRb and bRc, then cRa.



Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming

R is an equivalence relation.

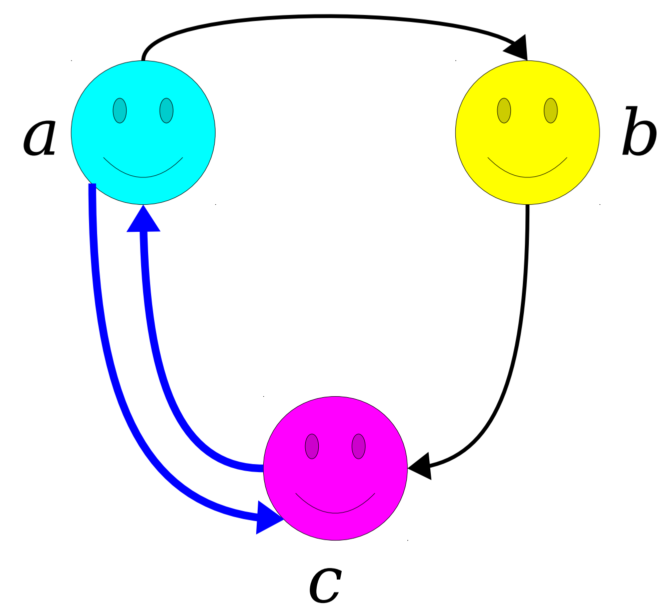
R is reflexive.

- R is symmetric.

R is transitive.

What We Need To Show

- If aRb and bRc, then cRa.



Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof:

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmet… already know that… that R is cyclic.

To prove that R is … A where aRb and bR… Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

Notice how the first few sentences of this proof mirror the structure of what needs to be proved. We’re just following the templates from the first week of class!

Lemma … en R
is refl…

Proof: L… me
set A. …

Since …
reflexi…
alread… ow
that R is cyclic.

Notice how this setup mirrors the first-order definition of cyclicity:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds.
Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

Lem… …en R is…

Although this proof is deeply informed by the first-order definitions, notice that there is no first-order logic notation anywhere in the proof. That's normal – it's actually quite rare to see first-order logic in written proofs.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
- R is cyclic.

What We Need To Show

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.

R is cyclic.

What We Need To Show

R is an equivalence relation.

- R is reflexive.

R is symmetric.

R is transitive.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
- R is cyclic.

What We Need To Show

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
- R is cyclic.

What We Need To Show

R is an equivalence relation.

- R is reflexive.
- R is symmetric.
- R is transitive.

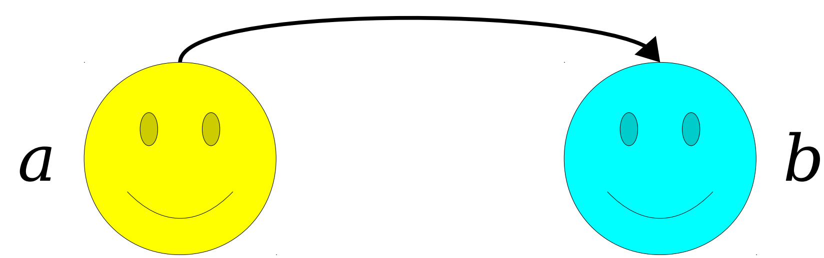
Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
- R is cyclic.

What We Need To Show

- R is symmetric.
  - If aRb, then bRa.



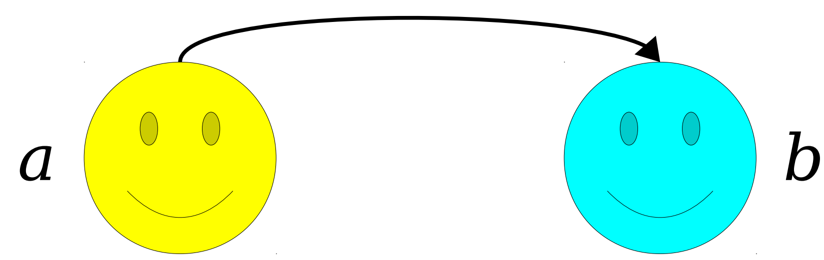
Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
  - $\forall x \in A.\ xRx$
- R is cyclic.
  - $xRy \land yRz \to zRx$

What We Need To Show

- R is symmetric.
  - If aRb, then bRa.



Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

R is reflexive.

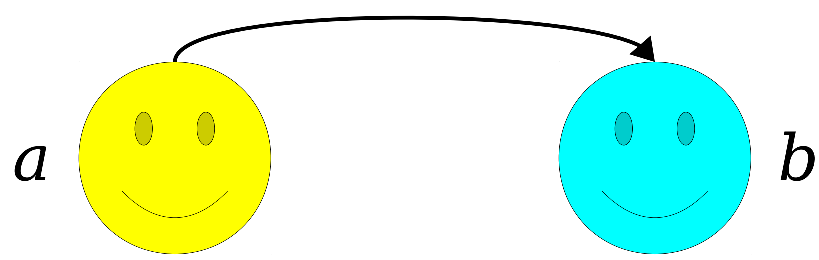
$\forall x \in A.\ xRx$

R is cyclic.

- $xRy \land yRz \rightarrow zRx$

What We Need To Show

- R is symmetric.
  - If aRb, then bRa.



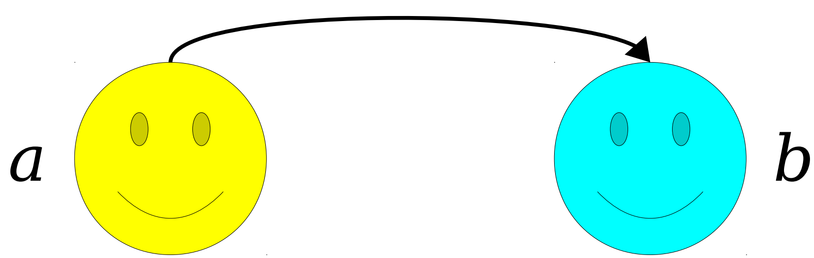
Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
  - $\forall x \in A.\ xRx$
- R is cyclic.
  - $xRy \land yRz \rightarrow zRx$

What We Need To Show

- R is symmetric.
  - If aRb, then bRa.



Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

R is reflexive.

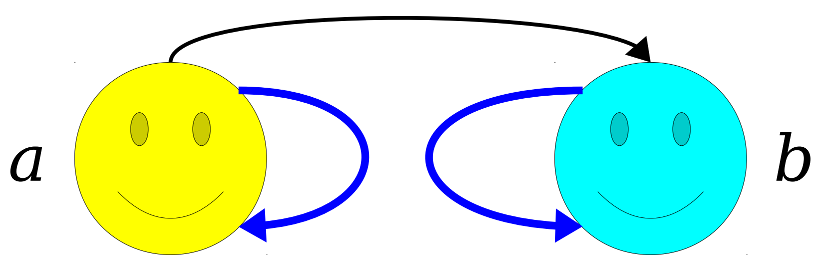
- $\forall x \in A.\ xRx$

R is cyclic.

$xRy \land yRz \to zRx$

What We Need To Show

- R is symmetric.
  - If aRb, then bRa.



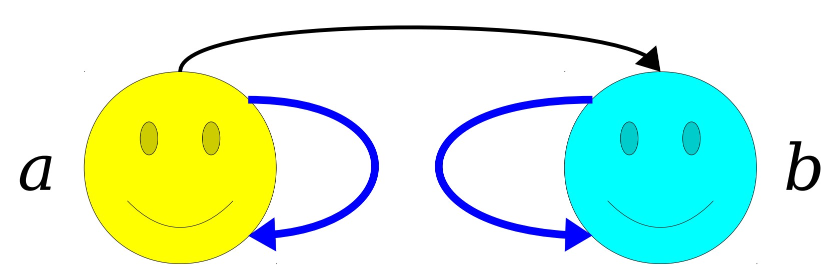
Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
  - $\forall x \in A.\ xRx$
- R is cyclic.
  - $xRy \land yRz \to zRx$

What We Need To Show

- R is symmetric.
  - If aRb, then bRa.



Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

R is reflexive.

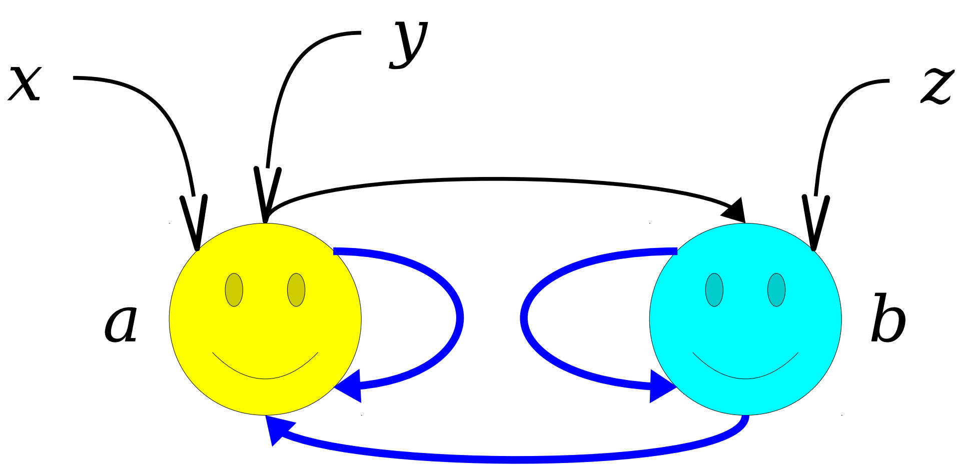
$\forall x \in A.\ xRx$

R is cyclic.

- $xRy \land yRz \rightarrow zRx$

What We Need To Show

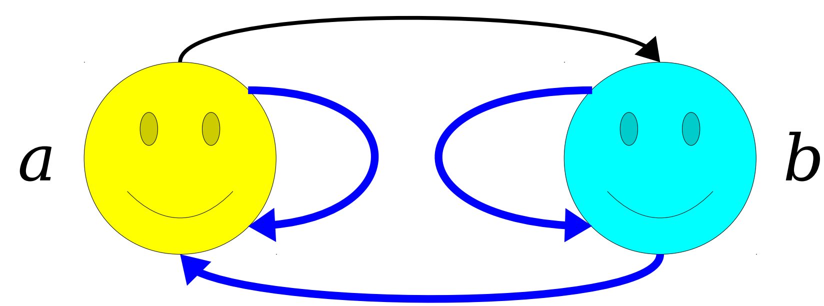
- R is symmetric.
  - If aRb, then bRa.



Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming	*What We Need To Show*

• R is reflexive.
  • $\forall x \in A.\ xRx$
• R is cyclic.
  • $xRy \land yRz \to zRx$
	• R is symmetric.
	  • If aRb, then bRa.



Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
  - $\forall x \in A.\ xRx$
- R is cyclic.
  - $xRy \land yRz \to zRx$

What We Need To Show

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
  - $\forall x \in A.\ xRx$
- R is cyclic.
  - $xRy \land yRz \rightarrow zRx$

What We Need To Show

R is an equivalence relation.

R is reflexive.

R is symmetric.

- R is transitive.

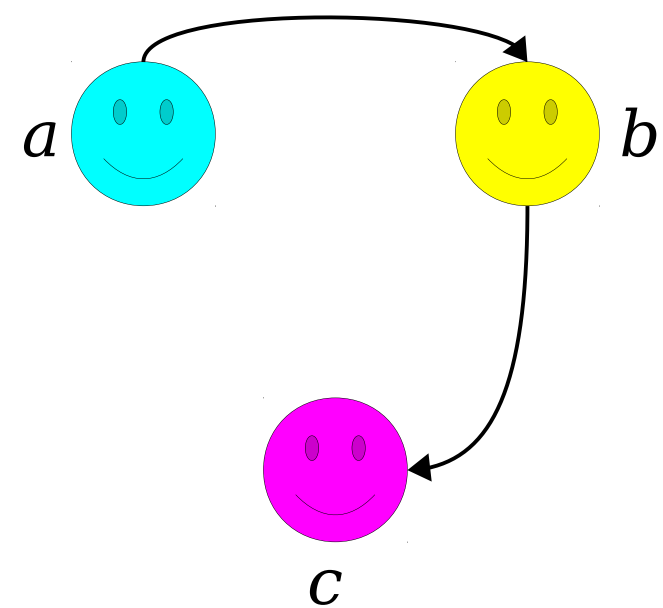
Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
  - $\forall x \in A.\ xRx$
- R is cyclic.
  - $xRy \land yRz \to zRx$

What We Need To Show

- R is transitive.
  - If aRb and bRc, then aRc.



Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

R is reflexive.

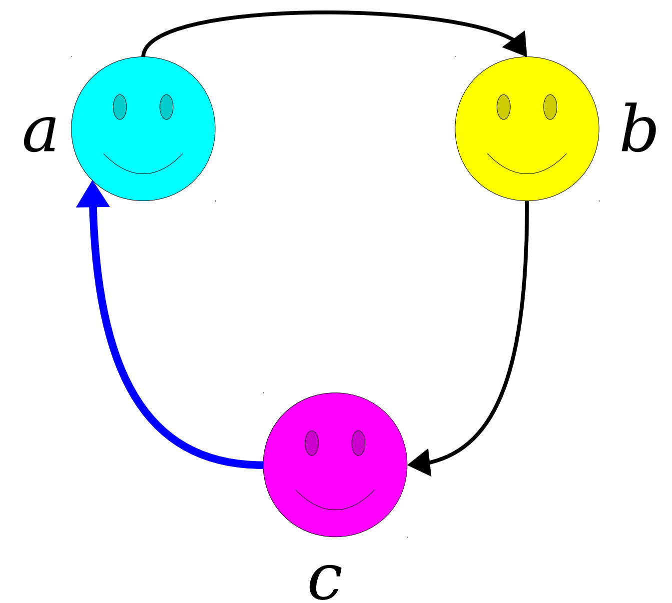
$\forall x \in A.\ xRx$

R is cyclic.

$xRy \land yRz \to zRx$

What We Need To Show

- R is transitive.
  - If aRb and bRc, then aRc.



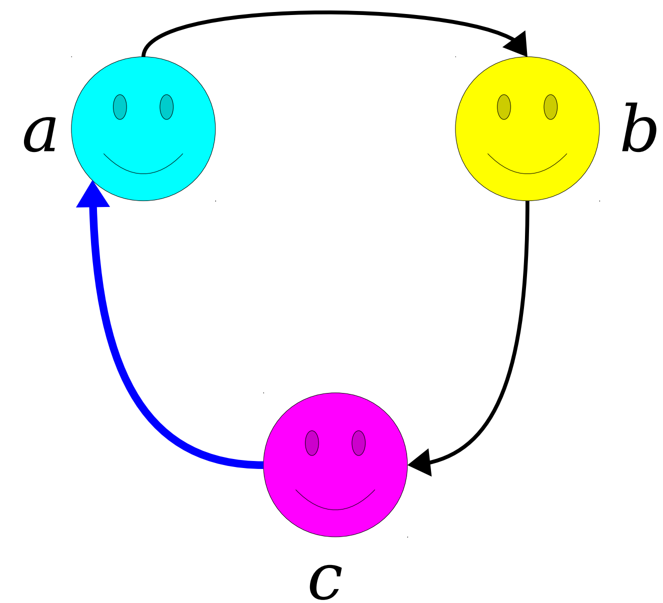
Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
  - $\forall x \in A.\ xRx$
- R is cyclic.
  - $xRy \land yRz \to zRx$

What We Need To Show

- R is transitive.
  - If aRb and bRc, then aRc.



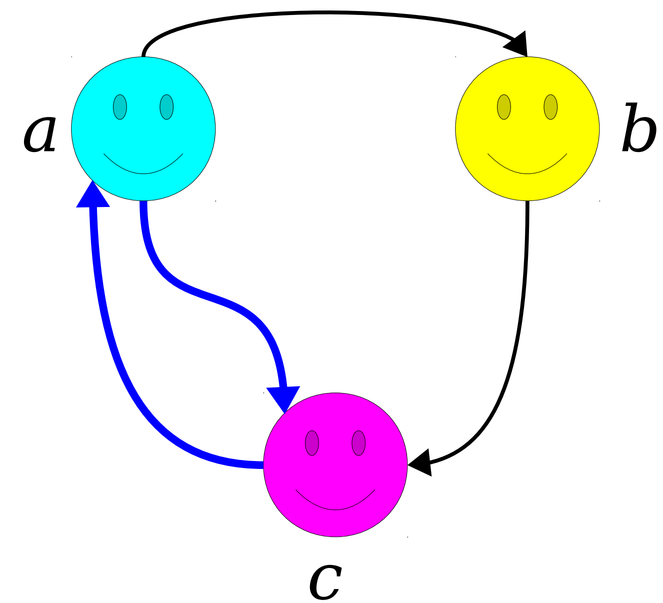
Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
  - $\forall x \in A.\ xRx$
- R is cyclic.
  - $xRy \land yRz \to zRx$
- R is symmetric
  - $xRy \to yRx$

What We Need To Show

- R is transitive.
  - If aRb and bRc, then aRc.



Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof:

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required.

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

Lemm ... clic
anc

Proof ... that
is c
equ ... R is
ref
kno ... o
sho

Notice how this setup mirrors the first-order definition of symmetry:

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equi … is refle … know … show …

First … y arbit … hat bRa … s. Then … t bRa, as required.

Notice how this setup mirrors the first-order definition of transitivity:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \rightarrow aRc)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

Refining Your Proofwriting

- When writing proofs about terms with formal definitions, you ***must*** call back to those definitions.
  - Use the first-order definition to see what you'll assume and what you'll need to prove.
- When writing proofs about terms with formal definitions, you ***must not*** include any first-order logic in your proofs.
  - Although you won't use any FOL *notation* in your proofs, your proof implicitly calls back to the FOL definitions.
- You'll get a lot of practice with this on Problem Set Three. If you have any questions about how to do this properly, please feel free to ask on Piazza or stop by office hours!

Next Time

- ***Functions***
  - How do we model transformations in a mathematical sense?
- ***Domains and Codomains***
  - Type theory meets mathematics!
- ***Injections, Surjections, and Bijections***
  - Three special classes of functions.